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On the electromagnetic properties of matter in collapse models

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Abstract

We discuss the electromagnetic properties of both a charged free particle, and a charged particle bounded by an harmonic potential, within collapse models. By choosing a particularly simple—yet physically relevant—collapse model, and under only the dipole approximation, we are able to solve the equation of motion exactly. In this way, both the finite-time and large-time behavior can be analyzed accurately. We discovered new features, which did not appear in previous works on the same subject. Since, so far, the spontaneous photon emission process places the strongest upper bounds on the collapse parameters, our results call for a further analysis of this process for those atomic systems which can be employed in experimental tests of collapse models, as well as of quantum mechanics.

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1. Introduction

Models of spontaneous wavefunction collapse [1–5] provide a description of quantum (and classical) phenomena, which is free of the much debated measurement problem affecting the standard quantum theory. This is achieved by modifying the Schrödinger equation, adding nonlinear and stochastic terms which reproduce, at a suitable scale, the process of wavefunction collapse.

By modifying the Schrödinger equation, these models make predictions which differ from those of standard quantum mechanics. It is interesting and important to analyze such differences, not only for a better understanding of these models, but also for deciding which experimental setups are more convenient, in order to test them against quantum mechanics. Such experiments, needless to say, would represent important tests also for the quantum theory itself [6].

It has emerged from the work reported in [7–11] that the electromagnetic properties of matter place, so far, the strongest upper bound on the collapse frequency λ_{GRW} of the GRW (Ghirardi–Rimini–Weber) model [1] (or, equivalently, the parameter γ of the CSL³ (continuous spontaneous localization) model [3]). More specifically, it has been proven that charged particles spontaneously emit radiation, as a consequence of the interaction with the collapsing field, also when according to standard quantum mechanics no radiation should be emitted; the radiation spectrum has been computed both for a free charged particle [10] and for an hydrogenic atom [11]. The theoretical spectrum has been compared with available experimental data, placing an upper bound [11] of only six orders of magnitude away from the standard CSL value⁴ $\gamma = 10^{-30} \text{ cm}^3 \text{ s}^{-1}$. Note that more direct experiments of the superposition principle of quantum mechanics, such as diffraction experiments with macromolecules [12, 13], place a much weaker upper bound, which is 13 orders of magnitude away from the standard CSL value [14]. These figures show that analyzing the electromagnetic properties of matter within collapse models is particularly relevant, not only *per se* but also in view of future experimental tests.

The above-mentioned analysis has been carried out to first order in perturbation theory, using the CSL model. The goal of this work is to deepen our understanding of the process of spontaneous photon emission from charged particles. We will do it by using, in place of the CSL model, the simpler QMUPL (quantum mechanics with universal position localizations) model [15, 16] and we will work under the dipole approximation. These assumptions will allow us to solve the equations of motion exactly: we will derive an exact formula for the spectrum of the emitted radiation, valid to all orders, and we will compare it with the formulas obtained in [10, 11]. As we will see, new features will emerge, previously not discussed.

The QMUPL model of spontaneous wavefunction collapse applies to systems of distinguishable non-relativistic particles. The one-particle equation, which is sufficient for the purposes of this paper, reads

$$d\psi_t = \left[-\frac{i}{\hbar} H dt + \sqrt{\lambda}(\mathbf{q} - \langle \mathbf{q} \rangle_t) d\mathbf{W}_t - \frac{\lambda}{2}(\mathbf{q} - \langle \mathbf{q} \rangle_t)^2 dt \right] \psi_t, \quad (1)$$

its generalization to a many-particle system being straightforward. In the above equation, H is the standard quantum Hamiltonian of the particle, \mathbf{q} is its position operator, \mathbf{W}_t are three independent standard Wiener processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and λ is a positive constant which sets the strength of the collapse mechanics. The physical content of the above equation is the following. The first term on the right-hand side gives the usual unitary evolution, driven by the Hamiltonian H . The second and third terms cause the collapse of the wavefunction toward a state which is localized in *space*, being driven by the *position* operator. More specifically (but not entirely correctly; for a discussion see [17]) the third term localizes the wavefunction—the more negative it is, the greater the difference $|\mathbf{q} - \langle \mathbf{q} \rangle_t|$ —while the second term, which contains the random process \mathbf{W}_t , ensures that the collapse occurs

³ The GRW and CSL models are the two most popular models of spontaneous wavefunction collapse. Their main difference is that the first assumes the collapses to occur at discrete random times, through a jump process, while the second assumes the collapse to occur continuously, through a diffusion process.

⁴ This value for the collapse parameter γ has been chosen in such a way that for a single constituent—in which case the GRW and CSL models coincide at the statistical level—the reduction occurs with the rate $\lambda_{\text{GRW}} \simeq 2.2 \times 10^{-17} \text{ s}^{-1}$ of the GRW model [1]. The relation between the two constants is $\lambda_{\text{GRW}} = \gamma(\alpha/4\pi)^{3/2}$, with $\alpha = 10^{10} \text{ cm}^{-2}$ [3]. This choice implies that the two models behave similarly, although important differences arise, due to the fact that the GRW model refers only to systems of distinguishable particles, while the CSL model takes into account also identical particles. The numerical value for λ_{GRW} was originally chosen in such a way to ensure that superpositions of macroscopic objects (containing roughly an Avogadro's number of constituents) are localized within the perception time of a human being, while microscopic systems retain all their quantum properties [1].

randomly and in agreement with the Born probability rule. The structure $(\mathbf{q} - \langle \mathbf{q} \rangle_t)$ ensures that the wavefunction remains normalized, even if the dynamics is not unitary anymore.

This model, in spite of its simplicity, is particularly relevant because, in an appropriate limit which we shall now briefly discuss, it reduces at the statistical level to the more familiar GRW model (thus also to the CSL model, as long as the particles are distinguishable). Here again we limit our consideration to one single particle. The master equation describing the time evolution of the statistical ensemble $\rho_t \equiv \mathbb{E}_{\mathbb{P}}[|\psi_t\rangle\langle\psi_t|]$, where ψ_t evolves according to equation (1), has a Lindblad form, which in the position representation (where $\rho_t(\mathbf{x}, \mathbf{y}) \equiv \langle \mathbf{x} | \rho_t | \mathbf{y} \rangle$) reads [16]

$$\frac{d}{dt} \rho_t(\mathbf{x}, \mathbf{y}) = -\frac{i}{\hbar} [H, \rho_t(\mathbf{x}, \mathbf{y})] - \frac{\lambda}{2} (\mathbf{x} - \mathbf{y})^2 \rho_t(\mathbf{x}, \mathbf{y}). \quad (2)$$

On the other hand, the one-particle GRW (and CSL) master equation reads [1]

$$\frac{d}{dt} \rho_t(\mathbf{x}, \mathbf{y}) = -\frac{i}{\hbar} [H, \rho_t(\mathbf{x}, \mathbf{y})] - \lambda_{\text{GRW}} [1 - e^{-\frac{\alpha}{4}(\mathbf{x}-\mathbf{y})^2}] \rho_t(\mathbf{x}, \mathbf{y}), \quad (3)$$

for the relation between the constant λ_{GRW} characterizing the GRW model and the constant γ defining the strength of the collapse process in the CSL model, see the previous footnote. The second parameter (α) defines a correlation length $r_C = 1/\sqrt{\alpha} \simeq 10^{-5}$ cm, above which spatial superpositions are reduced.

Let us now consider situations where, for all values of \mathbf{x} and \mathbf{y} such that the density matrix $\rho_t(\mathbf{x}, \mathbf{y})$ is appreciably different from 0, one has $|\mathbf{x} - \mathbf{y}| \ll r_C$. We call this the *small distances* assumption. This is the case if the physical system is localized well below r_C , as happens e.g. for atoms in matter. In this case, it makes sense to take, in equation (3), the limit $\alpha \rightarrow 0$ and $\lambda_{\text{GRW}} \rightarrow \infty$, while keeping the product $\lambda_{\text{GRW}}\alpha$ constant. Then, equation (3) reduces to (2), with the identification

$$\lambda = \frac{\alpha \lambda_{\text{GRW}}}{2} = \frac{\alpha^{3/2} \gamma}{16\pi^{3/2}}. \quad (4)$$

Accordingly, the QMUPL model represents, at the statistical level, a good approximation of the GRW models, for those systems which are well localized with respect to the correlation length r_C .

2. Motion of a charged particle interacting with the electromagnetic field, bounded by a linear force, subject to collapse in space

In this section we explicitly solve, under only the dipole approximation, the equations of motion for a non-relativistic charged particle interacting with the second quantized electromagnetic field. The particle is bounded by an harmonic potential—the limit case of a free particle will also be discussed—and is subject to spontaneous collapses in space according to the QMUPL model.

Equation (1) is nonlinear, but it can be appropriately reduced to a linear (though not norm-preserving) equation through a standard procedure [16]. Of course, non-linearity is not canceled; it reappears when the statistical properties (through a change of measure) are computed. However, since we are ultimately concerned only with physical quantities of the type $\mathbb{E}_{\mathbb{P}}[\langle \psi_t | O | \psi_t \rangle]$, where O is any suitable self-adjoint operator, we can use the following mathematical property. Consider the class of SDEs:

$$d\psi_t^\zeta = \left[-\frac{i}{\hbar} H dt + \sqrt{\lambda} (\zeta \mathbf{q} - \zeta_{\text{R}} \langle \mathbf{q} \rangle_t) d\mathbf{W}_t - \frac{1}{2} \lambda (|\zeta|^2 \mathbf{q}^2 - 2\zeta \zeta_{\text{R}} \langle \mathbf{q} \rangle_t + \zeta_{\text{R}}^2 \langle \mathbf{q} \rangle_t^2) dt \right] \psi_t^\zeta, \quad (5)$$

where ζ is a complex phase, and ζ_{R} its real part. A straightforward application of Itô calculus allows us to prove that $\mathbb{E}_{\mathbb{P}}[\langle \psi_t | O | \psi_t \rangle]$ is independent of ζ , in spite of the fact that equation (5)

describes completely different evolutions for the wavefunction, for different values of ζ . In particular, when $\zeta = 1$, equation (5) coincides with the QMUPL collapse equation (1). On the other hand, when $\zeta = i$, equation (5) reduces to the simpler SDE:

$$d\psi_t = \left[-\frac{i}{\hbar} H dt + i\sqrt{\lambda}\mathbf{q} d\mathbf{W}_t - \frac{1}{2}\lambda\mathbf{q}^2 dt \right] \psi_t, \quad (6)$$

where all the non-linear terms have disappeared. Such an equation of course does not lead to the collapse of the wavefunction, since it describes a linear and unitary⁵, though stochastic, evolution. Nevertheless, it is as good as equation (1) for computing average quantities. The advantage is that its linearity and unitarity make calculations easier.

Equation (6) has to be understood in the Itô sense. We will solve the corresponding Stratonovich equation, where the stochastic differential $d\mathbf{W}_t$ can be interpreted as the increment of a white noise $\mathbf{w}(t)$:

$$i\hbar \frac{d}{dt} \psi_t = [H - \sqrt{\lambda}\hbar \mathbf{q} \mathbf{w}(t)] \psi_t. \quad (7)$$

This is a standard Schrödinger equation with a random potential depending on the position \mathbf{q} of the particle. Note that the last term of equation (6) has disappeared in going from the Itô to the Stratonovich formulation of the SDE. Actually, one can be more general and assume that $\mathbf{w}(t)$ represents three Gaussian noises with zero mean and a general correlation function, without having to change the mathematical formalism. However, this goes beyond the scope of the present analysis, so we will keep assuming that $\mathbf{w}(t)$ is white noise.

Coming back to our physical system, the standard Hamiltonian H is

$$H = \frac{1}{2m_0} (\mathbf{p} - e\mathbf{A})^2 + \frac{1}{2}\kappa \mathbf{q}^2 + \frac{1}{2}\epsilon_0 \int d^3x [\mathbf{E}^2 + c^2\mathbf{B}^2], \quad (8)$$

where m_0 is the bare mass of the particle, κ is the force constant of the harmonic term, \mathbf{A} is the vector potential, \mathbf{E} and \mathbf{B} are the electric and magnetic fields, respectively, e is the electric charge, c is the speed of light and ϵ_0 is the vacuum permittivity. Throughout this section, we use the gauge $\nabla \cdot \mathbf{A} = 0$ and $V = 0$, where V is the electromagnetic scalar potential⁶.

The plane wave decomposition of the vector potential \mathbf{A} reads

$$\mathbf{A}(\mathbf{x}) = \sqrt{\frac{\hbar}{\epsilon_0}} \sum_{\mu} \int \frac{d^3k}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} [a_{\mathbf{k}\mu} e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}\mu}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{x}}], \quad (9)$$

where $\omega_k = ck$ ($k = |\mathbf{k}|$) is the frequency corresponding to the wave vector \mathbf{k} , $\epsilon_{\mathbf{k}\mu}$ ($\mu = 1, 2$) are the linear polarization vectors and $a_{\mathbf{k}\mu}^{\dagger}$, $a_{\mathbf{k}\mu}$ are the creation and annihilation operators, respectively, satisfying the standard commutation relations:

$$[a_{\mathbf{k}\mu}, a_{\mathbf{k}'\mu'}^{\dagger}] = \delta_{\mu\mu'} \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \quad (10)$$

Up to now the model is exact, but not exactly solvable. To further proceed in the analysis, we make the *dipole approximation* $e^{i\mathbf{k}\cdot\mathbf{x}} \simeq 1$, which holds as long as the wavelength of the electromagnetic radiation is much larger than the typical size of an atom. Note that this assumption is compatible with the *small distances* assumption discussed in the previous section. The resulting model turns out to be ultraviolet divergent: we cure this problem

⁵ The third term of equation (6) is an 'Itô term', which disappears from the solution of the equation. For this reason the evolution is unitary, even if apparently it does not look so.

⁶ Note that we are assuming that the spontaneous collapse process occurs only for the particle, not for the electromagnetic field. The reason is that, so far, collapse models have been considered only for massive particles, the localization of their wavefunction being sufficient for solving the measurement problem of quantum mechanics. However, in a more speculative scenario, e.g. where the collapse mechanism is linked to gravitational phenomena, one could assume that also the photons' wavefunction undergoes a spontaneous localization process.

by introducing a form factor $g(\mathbf{k})$, corresponding to the Fourier transform of the charge distribution (normalized to unity):

$$g(\mathbf{k}) := \int \frac{d^3k}{\sqrt{(2\pi)^3}} \rho(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}, \quad \int d^3r \rho(\mathbf{r}) = 1. \quad (11)$$

Under these approximations, the vector potential (9) becomes

$$\mathbf{A}(\mathbf{x}) = \sqrt{\frac{\hbar}{\epsilon_0}} \sum_{\mu} \int d^3k \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} [a_{\mathbf{k}\mu} + a_{\mathbf{k}\mu}^{\dagger}], \quad (12)$$

and the factor $(2\pi)^{-3/2}$ has been included in the definition of $g(\mathbf{k})$. In this way, in the point-particle limit, $g(\mathbf{k}) \rightarrow 1/\sqrt{(2\pi)^3}$.

Since the total Hamiltonian in equation (7) is a standard—though stochastic—Hamiltonian, one can conveniently work in the Heisenberg picture. The equations of motions for the position $\mathbf{q}(t)$ of the particle and of the conjugate momentum $\mathbf{p}(t)$ can be immediately derived:

$$\frac{d\mathbf{p}}{dt} = -\kappa \mathbf{q} + \sqrt{\lambda\hbar} \mathbf{w}(t), \quad (13)$$

$$\frac{d\mathbf{q}}{dt} = \frac{\mathbf{p}}{m_0} - \frac{e}{m_0} \mathbf{A}, \quad (14)$$

while the equation of motion for the electromagnetic field operator $a_{\mathbf{k}\mu}^{\dagger}(t)$ is

$$\begin{aligned} \frac{da_{\mathbf{k}\mu}^{\dagger}}{dt} &= i\omega_k a_{\mathbf{k}\mu}^{\dagger} - \frac{ie}{\sqrt{\hbar\epsilon_0}m_0} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \mathbf{p} \\ &+ \frac{ie^2}{\epsilon_0 m_0} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \sum_{\mu'} \int d^3k' \frac{g(\mathbf{k}')}{\sqrt{2\omega_{k'}}} \epsilon_{\mathbf{k}'\mu'} [a_{\mathbf{k}'\mu'} + a_{\mathbf{k}'\mu'}^{\dagger}] \end{aligned} \quad (15)$$

the equation for $a_{\mathbf{k}\mu}(t)$ can be obtained from the previous one by taking the Hermitian conjugate. The above set of coupled linear differential equations can be conveniently solved with the help of the Laplace transform; the equations for the transformed variables (which are denoted by a tilde) read

$$z\tilde{\mathbf{p}}(z) - \mathbf{p}(0) = -\kappa\tilde{\mathbf{q}}(z) + \hbar\sqrt{\lambda}\tilde{\mathbf{w}}(z), \quad (16)$$

$$z\tilde{\mathbf{q}}(z) - \mathbf{q}(0) = \frac{\tilde{\mathbf{p}}(z)}{m_0} - \frac{e}{m_0} \sqrt{\frac{\hbar}{\epsilon_0}} \sum_{\mu} \int d^3k \frac{1}{\sqrt{2\omega_k}} g(\mathbf{k}) \epsilon_{\mathbf{k}\mu} [\tilde{a}_{\mathbf{k}\mu}^{\dagger}(z) + \tilde{a}_{\mathbf{k}\mu}(z)], \quad (17)$$

$$z\tilde{a}_{\mathbf{k}\mu}^{\dagger}(z) - a_{\mathbf{k}\mu}^{\dagger}(0) = i\omega_k \tilde{a}_{\mathbf{k}\mu}^{\dagger}(z) - \frac{ie}{\sqrt{\hbar\epsilon_0}} \frac{1}{\sqrt{2\omega_k}} g(\mathbf{k}) \epsilon_{\mathbf{k}\mu} \cdot [z\tilde{\mathbf{q}}(z) - \mathbf{q}(0)], \quad (18)$$

$$z\tilde{a}_{\mathbf{k}\mu}(z) - a_{\mathbf{k}\mu}(0) = -i\omega_k \tilde{a}_{\mathbf{k}\mu}(z) + \frac{ie}{\sqrt{\hbar\epsilon_0}} \frac{1}{\sqrt{2\omega_k}} g(\mathbf{k}) \epsilon_{\mathbf{k}\mu} \cdot [z\tilde{\mathbf{q}}(z) - \mathbf{q}(0)], \quad (19)$$

where z is the transformed time. The above set now represents a system of coupled algebraic equations, which can be solved in a standard way. The calculation is long but straightforward; transforming back to the original variables, one obtains

$$\begin{aligned} \mathbf{q}(t) &= [1 - \kappa F_1(t)] \mathbf{q}(0) + F_0(t) \mathbf{p}(0) \\ &- e \sqrt{\frac{\hbar}{\epsilon_0}} \sum_{\mu} \int d^3k \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} [G_1^+(k, t) a_{\mathbf{k}\mu}(0) + G_1^-(k, t) a_{\mathbf{k}\mu}^{\dagger}(0)] \\ &+ \sqrt{\lambda\hbar} \int_0^t ds F_0(t-s) \mathbf{w}(s), \end{aligned} \quad (20)$$

$$\begin{aligned}
\mathbf{p}(t) = & -\kappa [t - \kappa F_2(t)] \mathbf{q}(0) + [1 - \kappa F_1(t)] \mathbf{p}(0) \\
& + \kappa e \sqrt{\frac{\hbar}{\epsilon_0}} \sum_{\mu} \int d^3k \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} [G_0^+(k, t) a_{\mathbf{k}\mu}(0) + G_0^-(k, t) a_{\mathbf{k}\mu}^\dagger(0)] \\
& + \sqrt{\lambda \hbar} \int_0^t ds [1 - \kappa F_1(t-s)] \mathbf{w}(s), \tag{21}
\end{aligned}$$

$$\begin{aligned}
a_{\mathbf{k}\mu}^\dagger(t) = & e^{-i\omega_k t} a_{\mathbf{k}\mu}^\dagger(0) - \frac{ie}{\sqrt{\hbar\epsilon_0}} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot [G_1^-(k, t) \mathbf{p}(0) - \kappa G_0^-(k, t) \mathbf{q}(0)] \\
& + \frac{ie^2}{\epsilon_0} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \sum_{\mu'} \int d^3k' \frac{g(\mathbf{k}')}{\sqrt{2\omega_{k'}}} \epsilon_{\mathbf{k}'\mu'} [G_+^-(k, k', t) a_{\mathbf{k}'\mu'}(0) \\
& + G_-^-(k, k', t) a_{\mathbf{k}'\mu'}^\dagger(0)] - ie \sqrt{\frac{\hbar\lambda}{\epsilon_0}} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \int_0^t ds G_1^-(k, t-s) \mathbf{w}(s), \tag{22}
\end{aligned}$$

$$\begin{aligned}
a_{\mathbf{k}\mu}(t) = & e^{-i\omega_k t} a_{\mathbf{k}\mu}(0) + \frac{ie}{\sqrt{\hbar\epsilon_0}} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot [G_1^+(k, t) \mathbf{p}(0) - \kappa G_0^+(k, t) \mathbf{q}(0)] \\
& - \frac{ie^2}{\epsilon_0} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \sum_{\mu'} \int d^3k' \frac{g(\mathbf{k}')}{\sqrt{2\omega_{k'}}} \epsilon_{\mathbf{k}'\mu'} [G_+^+(k, k', t) a_{\mathbf{k}'\mu'}(0) \\
& + G_-^+(k, k', t) a_{\mathbf{k}'\mu'}^\dagger(0)] + ie \sqrt{\frac{\hbar\lambda}{\epsilon_0}} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \int_0^t ds G_1^+(k, t-s) \mathbf{w}(s). \tag{23}
\end{aligned}$$

In the previous formulas, we have introduced the following functions:

$$F_n(t) = \int_{\Gamma} \frac{dz}{2\pi i} \frac{e^{zt}}{z^n H(z)}, \quad n = 0, 1, 2, \tag{24}$$

$$G_n^{\pm}(k, t) = \int_{\Gamma} \frac{dz}{2\pi i} \frac{z^n e^{zt}}{(z \pm i\omega_k) H(z)}, \quad n = 0, 1, \tag{25}$$

$$G_{\pm}^{\pm}(k, k', t) = \int_{\Gamma} \frac{dz}{2\pi i} \frac{z^2 e^{zt}}{(z \pm i\omega_k)(z \pm i\omega_{k'}) H(z)}; \tag{26}$$

in the third expression, the upper \pm refers to the first parenthesis, while the lower one refers to the second parenthesis. In all the above formulas, according to the theory of Laplace transform, the contour Γ must be a line parallel to the imaginary axis, lying to the right of all singularities of the integrand. The above solutions should be compared with those obtained in [18, 19], where the collapse process was not taken into account: apart from a marginal calculational mistake in [19] in the evolution of $\mathbf{p}(t)$, the two results agree when λ is set to zero in equations (20)–(23).

The function $H(z)$ is defined as follows:

$$H(z) = \kappa + z^2 \left[m_0 + \frac{8\pi e^2}{3\epsilon_0} \int_0^\infty dk g(k)^2 \frac{k^2}{z^2 + \omega_k^2} \right] \tag{27}$$

(from now on we assume the form factor to depend only on the modulus k of \mathbf{k}). This is a crucial quantity, as through formulas (24)–(26) it determines the time evolution of all physical quantities. It depends on the form factor $g(k)$: simply removing it, would make the integral

ultraviolet divergent. To overcome the problem, we apply a renormalization procedure. The quantity within square brackets in (27) can be rewritten as follows:

$$m_0 + \frac{8\pi e^2}{3\epsilon_0} \int_0^\infty dk g(k)^2 \frac{k^2}{z^2 + \omega_k^2} = \left(m_0 + \frac{4}{3}m_r\right) - \frac{8\pi e^2}{3\epsilon_0 c^2} z^2 \int_0^\infty dk \frac{g(k)^2}{z^2 + \omega_k^2}, \quad (28)$$

where m_r is the electrostatic mass:

$$m_r := \frac{e^2}{8\pi\epsilon_0 c^2} \int d^3r d^3r' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \equiv \frac{2\pi e^2}{\epsilon_0 c^2} \int d^3k g(k)^2. \quad (29)$$

When $g(\mathbf{k}) \rightarrow 1/\sqrt{(2\pi)^3}$, m_r diverges. We apply the classical renormalization procedure⁷ for a non-relativistic charged particle coupled to the electromagnetic field [20–22] (which is valid both as a classical calculation and as a Heisenberg picture, quantum mechanical one, as in our case). According to it, as $m_r \rightarrow +\infty$ in the point-particle limit, one assumes that $m_0 \rightarrow -\infty$, in such a way that $m := m_0 + (4/3)m_r$ remains finite. This is assumed to be the renormalized mass.

The last term of (28) instead remains finite in the limit, the integral can be evaluated, and $H(z)$ takes the well-behaved expression:

$$H(z) = \kappa + z^2 [m - \beta z], \quad \beta = \frac{e^2}{6\pi\epsilon_0 c^3} \simeq 5.71 \times 10^{-54} \text{ kg s}. \quad (30)$$

Note that β is precisely the coefficient in front of the Abraham–Lorentz force, which is responsible for the runaway behavior of the corresponding Abraham–Lorentz equation, as we shall soon see. $H(z)$ is a polynomial of third degree, whose zeros can be found by the standard Cardan method. One solution is real and two are complex conjugate. Let $\omega_0 := \sqrt{\kappa/m}$ be the frequency of the oscillator. By assuming $\omega_0 \ll 2m/\sqrt{27}\beta \simeq 6.14 \times 10^{22} \text{ s}^{-1}$ for an electron ($\hbar\omega_0 \ll 4.04 \times 10^4 \text{ keV}$), their approximate value is (see appendix A)

$$z_1 \simeq \frac{m}{\beta} + o(\omega_0), \quad z_{2,3} \simeq -\frac{\omega_0^2 \beta}{2m} \pm i\omega_0 + o(\omega_0^3). \quad (31)$$

Given the above results, the functions $F_n(t)$ and $G_n^\pm(k, t)$ defined in (24) and (25), which are the only ones we will use in the subsequent analysis, become

$$F_n(t) = \sum_{\ell=1}^3 z_\ell e^{z_\ell t} \left[\frac{z - z_\ell}{H(z)} \right]_{z=z_\ell} + \begin{cases} 0 & n = 0, \\ \omega_0^{-2} & n = 1, \\ t\omega_0^{-2} & n = 2, \end{cases} \quad (32)$$

$$G_n^\pm(k, t) = \sum_{\ell=1}^3 \frac{z_\ell^n e^{z_\ell t}}{(z_\ell \pm ik)} \left[\frac{z - z_\ell}{H(z)} \right]_{z=z_\ell} + \frac{(\pm ik)^n e^{\pm ikt}}{H(\pm ik)}. \quad (33)$$

The term in (32) and (33) with $\ell = 1$ diverges exponentially, since z_1 is positive. As we have anticipated, this is a manifestation of the runaway behavior of the Abraham–Lorentz equation [20–23]. In particular, in the free particle case ($\omega_0 = 0$), the coefficient z_1 corresponds to the rate of exponential growth of the acceleration, as discussed in textbooks. This problem is still open, and we pragmatically dismiss it by ignoring, in the subsequent formulas, all terms with $\ell = 1$.

⁷ One can note that the collapse terms do not enter the following equations; thus, the renormalization procedure applies as in standard cases.

2.1. The spectrum of the spontaneously emitted radiation

We are now in a position to compute the spectrum of the radiation spontaneously emitted by the particle, due to the interaction with the noise. Let $N_{\mathbf{k}\mu}(t) := a_{\mathbf{k}\mu}^\dagger(t)a_{\mathbf{k}\mu}(t)$ be the density of photons of wave vector \mathbf{k} and polarization μ . Let $|\phi\rangle := |\psi_{\text{ho}}\rangle|\Omega\rangle$ be the initial state of the system, where $|\psi_{\text{ho}}\rangle$ is the initial state of the harmonic oscillator and $|\Omega\rangle$ is the vacuum state for the electromagnetic field. Let finally $S(\mathbf{k}, \mu, t) := \mathbb{E}_{\mathbb{P}}[\langle\phi|N_{\mathbf{k}\mu}(t)|\phi\rangle]$ be the spectrum of the emitted radiation, averaged over the noise. By inspecting equations (22) and (23), one can note that all terms of $N_{\mathbf{k}\mu}(t)$ containing either $a_{\mathbf{k}\mu}^\dagger(0)$ or $a_{\mathbf{k}\mu}(0)$ give a zero contribution, when averaged with respect to the vacuum state, while all terms containing $\mathbf{w}(t)$ give a zero contribution, when the stochastic average is taken. Accordingly, $S(\mathbf{k}, \mu, t)$ is the sum of two terms:

$$S(\mathbf{k}, \mu, t) = S_{\text{qm}}(\mathbf{k}, \mu, t) + S_{\text{col}}(\mathbf{k}, \mu, t), \quad (34)$$

where $S_{\text{qm}}(\mathbf{k}, \mu, t)$ is the standard quantum formula, while $S_{\text{col}}(\mathbf{k}, \mu, t)$ is the contribution due to the noise. We are interested in computing this second term, which reads

$$S_{\text{col}}(\mathbf{k}, \mu, t) = \frac{\lambda\hbar e^2}{16\pi^3\epsilon_0} \frac{1}{\omega_k} \int_0^t ds G_1^-(k, t-s) G_1^+(k, t-s). \quad (35)$$

This is the main formula. In the next section, we will apply it to the two interesting cases of a free particle ($\omega_0 = 0$) and of a bounded particle ($\omega_0 \neq 0$).

3. The free particle

The free particle evolution can be deduced from the previous formulas by taking the limit $\omega_0 \rightarrow 0$. However, it turns out to be easier to redo the calculation, starting from equations (13)–(15) with $\kappa = 0$. The final result is

$$\begin{aligned} \mathbf{q}(t) = & \mathbf{q}(0) + \bar{F}_0(t)\mathbf{p}(0) - e\sqrt{\frac{\hbar}{\epsilon_0}} \sum_{\mu} \int d^3k \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} [\bar{G}_1^+(k, t) a_{\mathbf{k}\mu}(0) + \bar{G}_1^-(k, t) a_{\mathbf{k}\mu}^\dagger(0)] \\ & + \sqrt{\lambda\hbar} \int_0^t ds \bar{F}_0(t-s)\mathbf{w}(s), \end{aligned} \quad (36)$$

$$\mathbf{p}(t) = \mathbf{p}(0) + \sqrt{\lambda\hbar} \int_0^t ds \mathbf{w}(s), \quad (37)$$

$$\begin{aligned} a_{\mathbf{k}\mu}^\dagger(t) = & e^{i\omega_k t} a_{\mathbf{k}\mu}^\dagger(0) - \frac{ie}{\sqrt{\hbar\epsilon_0}} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \bar{G}_1^-(k, t) \epsilon_{\mathbf{k}\mu} \cdot \mathbf{p}(0) \\ & + \frac{ie^2}{\epsilon_0} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \sum_{\mu'} \int d^3k' \frac{g(\mathbf{k}')}{\sqrt{2\omega_{k'}}} \epsilon_{\mathbf{k}'\mu'} [\bar{G}_+^-(k, k', t) a_{\mathbf{k}'\mu'}(0) \\ & + \bar{G}_-^-(k, k', t) a_{\mathbf{k}'\mu'}^\dagger(0)] - ie\sqrt{\frac{\hbar\lambda}{\epsilon_0}} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \int_0^t ds \bar{G}_1^-(k, t-s)\mathbf{w}(s), \end{aligned} \quad (38)$$

$$\begin{aligned} a_{\mathbf{k}\mu}(t) = & e^{-i\omega_k t} a_{\mathbf{k}\mu}(0) + \frac{ie}{\sqrt{\hbar\epsilon_0}} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \bar{G}_1^+(k, t) \epsilon_{\mathbf{k}\mu} \cdot \mathbf{p}(0) \\ & - \frac{ie^2}{\epsilon_0} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \sum_{\mu'} \int d^3k' \frac{g(\mathbf{k}')}{\sqrt{2\omega_{k'}}} \epsilon_{\mathbf{k}'\mu'} [\bar{G}_+^+(k, k', t) a_{\mathbf{k}'\mu'}(0) \\ & + \bar{G}_-^+(k, k', t) a_{\mathbf{k}'\mu'}^\dagger(0)] + ie\sqrt{\frac{\hbar\lambda}{\epsilon_0}} \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \epsilon_{\mathbf{k}\mu} \cdot \int_0^t ds \bar{G}_1^+(k, t-s)\mathbf{w}(s), \end{aligned} \quad (39)$$

with

$$\bar{F}_0(t) = \frac{t}{m} + \frac{\beta^2}{m^2} e^{mt/\beta}, \quad (40)$$

$$\bar{G}_1^\pm(k, t) = \mp \frac{i}{m\omega_k} \mp \frac{i e^{\mp i\omega_k t}}{\omega_k(m \pm i\omega_k)} + \frac{e^{mt/\beta}}{(m/\beta)[(m/\beta) \pm i\omega_k]}, \quad (41)$$

$$\begin{aligned} \bar{G}_\pm^\pm(k, k', t) &= \frac{e^{\mp i\omega_k t}}{i(\mp\omega_k \pm \omega_{k'})(m \pm i\beta\omega_k)} + \frac{e^{\mp i\omega_{k'} t}}{i(\mp\omega_{k'} \pm \omega_k)(m \pm i\beta\omega_{k'})} \\ &+ \frac{e^{mt/\beta}}{[(m/\beta) \pm i\omega_k][(m/\beta) \pm i\omega_{k'}]}. \end{aligned} \quad (42)$$

In the last expression, the upper \pm refers to the sign in front of each ω_k , while the lower \pm refers to the sign in front of each $\omega_{k'}$. Once again, in all the above formulas we have a run-away behavior, as a consequence of the renormalization procedure. In the subsequent analysis, we neglect such terms.

There are two quantities which are of particular interest, in order to understand the behavior of the free charged particle under the influence of the collapsing field: the evolution of the mean kinetic energy, and the spectrum of the emitted radiation. We shall now discuss both of them.

3.1. The mean free kinetic energy

The mean kinetic energy of the particle is given by

$$E_{\text{mean}}(t) \equiv \frac{1}{2} m \mathbb{E}[\langle \phi | \dot{\mathbf{q}}(t)^2 | \phi \rangle]. \quad (43)$$

From equations (36) and (40) we have

$$\begin{aligned} \dot{\mathbf{q}}(t) &= \frac{\mathbf{p}(0)}{m} - e \sqrt{\frac{\hbar}{\epsilon_0}} \sum_\mu \int d^3k \frac{g(\mathbf{k})}{\sqrt{2\omega_k}} \boldsymbol{\epsilon}_{\mathbf{k}\mu} \left[\frac{e^{-i\omega_k t}}{m + i\beta\omega_k} a_{\mathbf{k}\mu}(0) + \frac{e^{i\omega_k t}}{m - i\beta\omega_k} a_{\mathbf{k}\mu}^\dagger(0) \right] \\ &+ \frac{\sqrt{\lambda\hbar}}{m} \int_0^t ds \mathbf{w}(s). \end{aligned} \quad (44)$$

By taking as initial state $|\phi\rangle = |\psi_{\text{free}}\rangle |\Omega\rangle$, as in the previous section, and after differentiating over time, one obtains the following expression:

$$\frac{d}{dt} E_{\text{mean}} = \frac{3}{2} \frac{\lambda\hbar^2}{m} = \frac{3}{4} \frac{\lambda_{\text{GRW}} \alpha_{\text{GRW}} \hbar^2}{m}, \quad (45)$$

which corresponds to the standard GRW formula [1]. We have then a very interesting result: in spite of the fact that—as we shall see in the next subsection—the particle emits radiation at a constant rate, its mean kinetic energy increases steadily in time as if the particle were neutral. In other words, the noise drives enough energy into the particle both to increase its kinetic energy and to make it radiate. This is a consequence of the fact that the collapse terms contain only the position operator \mathbf{q} , due to which \mathbf{w}_t acts like an *infinite temperature* noise; this feature was first pointed out in [24]. In the same reference, it was shown that a term proportional to the momentum operator acts like a dissipative term, thanks to which the mean energy thermalizes to a finite value, associated with a temperature which can be considered as the temperature of the noise. This is similar to what happens in the theory of quantum Brownian motion [25–27], and more generally in the theory of open quantum systems, which

does not come as a surprise, since collapse models and open quantum systems rely on similar master equations.

The above results can be read in two different ways. On a more conservative level, one can accept this steady energy increase as a feature of the model, as long as it does not violate known experimental data. On a more speculative level, it suggests that the coupling between the noise and the wavefunction should be modified in order for the total energy (energy of the noise, plus kinetic energy of the particle, plus energy of the emitted radiation) to be conserved. According to this view, the models so far proposed (GRW, CSL, QMUPL) are first approximations of more realistic models of spontaneous wavefunction collapse, yet to be formulated.

3.2. The spectrum of the emitted radiation

By using equation (35), with $G_1^\pm(k, t)$ given by equation (41), we obtain the following expression for the time derivative of the emitted spectrum:

$$\frac{d}{dt} S_{\text{col}}(\mathbf{k}, \mu, t) = \frac{\lambda \hbar e^2}{16\pi^3 \epsilon_0} \frac{1}{\omega_k} \left[\frac{2m^2 + \beta^2 \omega_k^2}{m^2 \omega_k^2 (m^2 + \beta^2 \omega_k^2)} + \frac{2\beta}{m \omega_k^2 (m^2 + \beta^2 \omega_k^2)} \omega_k \sin \omega_k t - \frac{2}{\omega_k^3 (m^2 + \beta^2 \omega_k^2)} \omega_k \cos \omega_k t \right]. \quad (46)$$

Since all observations are made over a period of time [10] much longer than the characteristic photon's frequencies, the two oscillating terms in the above expression average to 0. We are then left with only the first expression within brackets.

The physically interesting quantity is the spontaneous photon-emission rate $d\Gamma_k/dk$ per unit photon momentum. This is obtained from $dS_{\text{col}}(\mathbf{k}, \mu, t)/dt$ by summing over the polarization states and integrating over all directions in the photon's momentum space. The final result is

$$\frac{d}{dk} \Gamma_k = \frac{\lambda \hbar e^2}{2\pi^2 \epsilon_0 m^2 c^3 k} \cdot \frac{2 + (\beta ck/m)^2}{1 + (\beta ck/m)^2}. \quad (47)$$

It reassembles equation (21) of [9] (and equation (3.14) of [8]), when replacing $\epsilon_0 \rightarrow 1/4\pi$ because of the different system of units used, and when taking $\lambda = (m/m_N)^2 \lambda_0$ (m_N is the nucleon mass) as assumed in the mass-dependent CSL model [7]. The only difference is the extra factor $[2 + (\beta ck/m)^2]/[1 + (\beta ck/m)^2]$, the β dependence in which comes about because the result of [8, 9] has been carried out only to first perturbative order, while our result is exact (within the dipole approximation). For an electron, $(\beta ck/m)^2 \simeq (9.47 \times 10^{-6} E_k/\text{keV})^2$, where $E_k = \hbar ck$ is the energy of a photon of momentum k . Table 1 of [8] reports data from photons in an energy range between 11 and 501 keV: our calculation shows that, in this range, the first-order perturbation theory is extremely accurate.

Since equation (47) is valid for finite times, it provides a trustable understanding of the radiation process within the limits of the dipole approximation, i.e. as long as the particle does not move too fast, or as long as the photon's momentum is not too large. By keeping only the leading terms in the relevant parameters, i.e. by setting $\beta = 0$, equation (47) reduces to twice the large-time, first-order CSL expression of [10] and [11]. However, according to the argument of section 1, the CSL and QMUPL models should agree for sufficiently well-localized systems⁸ (with respect to the scale set by $r_C \simeq 10^{-5}$ cm); the origin of this discrepancy will be the subject of further exploration.

⁸ One can argue that the free particle case contradicts this assumption, as the wavefunction of a free particle rapidly spreads out in space; however, at least for sufficiently short times the approximation is correct.

As a last comment, we note that equation (47) predicts an infinite amount of energy to be emitted per unit time, as $d\Gamma_k/dk$ is of order $1/k$ for large k . This ultraviolet catastrophe is a consequence of the dipole approximation. One of the effects of the term $e^{ik \cdot x}$ in equation (9) is to temper the electromagnetic coupling for high frequencies, by replacing $e^{ik \cdot x}$ with 1, this effect is neglected. Accordingly, equation (47) is not trustable anymore in the very large k limit.

4. The harmonic oscillator

When the particle is bounded by a linear force, the emitted spectrum takes a quite different expression. By inserting equations (33), and ignoring the term $\ell = 1$ which gives a runaway solution, one finds

$$\begin{aligned}
 & \int_0^t G_1^-(k, t-s) G_1^+(k, t-s) \\
 &= \sum_{\ell, \ell'=2}^3 \frac{z_\ell z_{\ell'}}{(z_\ell + z_{\ell'})(z_\ell - i\omega_k)(z_{\ell'} + i\omega_k)} \\
 & \times \left[\frac{z - z_\ell}{H(z)} \right]_{z=z_\ell} \left[\frac{z - z_{\ell'}}{H(z)} \right]_{z=z_{\ell'}} [e^{(z_\ell + z_{\ell'})t} - 1] \\
 & - i \sum_{\ell=2}^3 \frac{\omega_k z_\ell}{H(-i\omega_k)(z_\ell - i\omega_k)^2} \left[\frac{z - z_\ell}{H(z)} \right]_{z=z_\ell} [e^{(z_\ell - i\omega_k)t} - 1] \\
 & + i \sum_{\ell=2}^3 \frac{\omega_k z_\ell}{H(+i\omega_k)(z_\ell + i\omega_k)^2} \left[\frac{z - z_\ell}{H(z)} \right]_{z=z_\ell} [e^{(z_\ell + i\omega_k)t} - 1] \\
 & + \frac{\omega_k^2}{H(-i\omega_k)H(+i\omega_k)} t. \tag{48}
 \end{aligned}$$

The formula is rather cumbersome. However, the terms in the first three lines contain exponentially decaying terms, which vanish very rapidly with time. For example—with reference to equation (31)—the decay time is about 2.93×10^{-47} s for an 11 keV photon. Accordingly, in the *large time limit* we have for the differential photon emission rate $d\Gamma_k/dk$ (where, as in the free particle case, we have differentiated equation (35) over time, summed over the polarization states and integrated over all directions in the photon's momentum space) the following simple *large-time* expression:

$$\frac{d\Gamma_k}{dk} = \frac{\lambda \hbar c e^2}{2\pi^2 \epsilon_0} \frac{k^3}{m^2 (\omega_0^2 - c^2 k^2)^2 + \beta^2 c^6 k^6}. \tag{49}$$

Two comments are at order. The first important thing one notes is that equation (49) does *not* reduce to (47) in the free particle limit. The reason for this incongruence can be traced back to equation (48), according to which the free particle limit ($\omega_0 \rightarrow 0$) and the large time limit ($t \rightarrow +\infty$) do not commute, as one can prove by direct calculation. From the physical point of view, the reason for the discrepancy is that, in the large time limit, the particle has the chance to move far enough to feel the edges of the harmonic potential, no matter how weak the potential is. This means that the particle is never really free, even in the limit $\omega_0 \rightarrow 0$. As a further proof of this statement, one can note that by taking the free particle limit at *finite* times, one does indeed recover equation (47). As a second observation, one can see that in the lowest order in the relevant parameters ($\beta = 0$), the emission rate given by equation (49)

is of order $1/k$ for $ck \gg \omega_0$. This is reminiscent of the free particle case. However, the exact expression is of order $1/k^3$, and the total emission is finite, contrary to what is implied by the free particle expression. The physical reason is that the binding potential works against the emission of high-energy photons, as the term $e^{ik \cdot x}$ in equation (9), which is neglected by the dipole approximation, does.

The third relevant observation is that equation (49) shows resonant behavior corresponding to the natural frequency ω_0 of the oscillator. Indeed the peak of the resonance is very high, due to the fact that $\beta^2 c^6 k^6$ is a very small quantity (compare the small value of β given in equation (30)) for $k = \omega_0/c$, where ω_0 is a standard frequency such as that associated with the hydrogen atom. Indeed such a great resonance is incompatible with experimental data and, as such, it would disprove this model, for any significant value of the collapse parameter λ . However, the large value of the peak is an artificial feature of the model. It emerges as a combination both of the fact that the energy levels of the harmonic oscillator are equally spaced, and from the dipole approximation, according to which transitions are allowed only between two consecutive levels. In other words, what happens here is that the noise excites the particle to a higher energy level state; in the de-excitation process only photons of energy $\hbar\omega_0$ can be emitted. In a more realistic model, also photons with any energy $n\hbar\omega_0$ should be emitted, and the spectrum would have a more articulated resonance structure, where the peaks are less pronounced. An accurate spectrum would then display several resonances.

To conclude, our analysis shows that, in the presence of a discrete spectrum (e.g. the hydrogen atom), the differential photon emission rate due to the collapse process should show typical resonant behavior, which has not been depicted by previous analysis. Although it is reasonable to expect that these resonances are highly suppressed, it is worthwhile analyzing such behavior for the CSL model, by generalizing the previous results of [10, 11] to the low-frequency part of the spectrum.

5. Conclusions

We have analyzed the electromagnetic properties of both a free particle and of a particle bounded by an harmonic potential, within the framework of collapse models. By choosing a particularly simple, yet physically meaningful, model of spontaneous wavefunction collapse, and under only the dipole approximation, we have been able to solve the equations of motion exactly.

In the free particle case, we have found a counterintuitive result: the particle's kinetic energy steadily increases in time, and at the same time it spontaneously emits radiation at a constant rate. Although this is in principle possible, as long as no conflict with experimental data emerges, such behavior suggests that collapse models should be modified in order to temper (or eliminate entirely) the evident violation of the energy conservation principle.

We have also found some discrepancies between our formula and those previously derived, through a perturbative analysis. The origin of these differences is not clear yet, and will be further studied in the future.

In the case of a particle confined by an harmonic potential, the spectrum is modified and a peak emerges, in correspondence to the natural frequency of the oscillator. This feature suggests that also in more realistic situations (e.g. atomic systems) the spectrum should have a resonant structure, which is worthwhile analyzing.

These results show that further analysis is required in order to better understand the electromagnetic properties of charged particles in the CSL model. This is important both for clarifying the theoretical picture offered by collapse models, and also in the light of future experimental tests.

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Appendix A. Derivation of equation (31)

The zeros of $H(z)$ defined by equation (30) correspond to the solution of the cubic equation:

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0, \quad a_0 = -\frac{\kappa}{\beta}, \quad a_1 = 0, \quad a_2 = -\frac{m}{\beta}. \quad (\text{A.1})$$

By defining

$$q = \frac{1}{3} a_1 - \frac{1}{9} a_2^2 \quad r = \frac{1}{6} (a_1 a_2 - 3a_0) - \frac{1}{27} a_2^3, \quad s_{1,2} = \sqrt[3]{r \pm \sqrt{q^3 + r^2}}, \quad (\text{A.2})$$

the three roots can be written as

$$z_1 = (s_1 + s_2) - \frac{1}{3} a_2, \quad z_{2,3} = -\frac{1}{2} (s_1 + s_2) - \frac{1}{3} a_2 \pm i \frac{\sqrt{3}}{2} (s_1 - s_2). \quad (\text{A.3})$$

This is the standard Cardan's method for finding the roots. In our case,

$$q^3 + r^2 = \frac{1}{4} a_0^2 \left(1 + \frac{4}{27} \frac{a_2^3}{a_0}\right) = \frac{1}{4} \frac{\kappa^2}{\beta^2} \left(1 + \frac{4}{27} \frac{m^2}{\omega_0^2 \beta^2}\right) \simeq \frac{1}{27} \frac{\omega_0^2 m^4}{\beta^4} \quad (\text{A.4})$$

if $\omega_0 \ll 2m/\sqrt{27}\beta$, as we have originally assumed. Then $\sqrt{q^3 + r^2} \simeq \omega_0 m^2 / \sqrt{27} \beta^2$. Working under the same approximation we have

$$r \pm \sqrt{q^3 + r^2} \simeq \frac{1}{27} \frac{m^3}{\beta^3} \left(1 \pm \sqrt{27} \frac{\omega_0 \beta}{m} + \frac{27}{2} \frac{\omega_0^2 \beta^2}{m^2}\right) \quad (\text{A.5})$$

and

$$s_{1,2} = \sqrt[3]{r \pm \sqrt{q^3 + r^2}} \simeq \frac{1}{3} \frac{m}{\beta} \left(1 \pm \sqrt{3} \frac{\omega_0 \beta}{m} + \frac{3}{2} \frac{\omega_0^2 \beta^2}{m^2}\right). \quad (\text{A.6})$$

From the above expression and from equation (A.3), the approximate values of the roots given in (31) can be immediately derived.

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